Received: *December 01, 2000*, Accepted: *January 04, 2001*

Remarks on twisted theories with matter

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Abstract: We investigate some aspects of $\mathcal{N}=2$ twisted theories with matter hypermultiplets in the fundamental representation of the gauge group. A consistent formulation of these theories on a general four-manifold requires turning on a particular magnetic flux, which we write down explicitly in the case of $\mathrm{SU}(2\ell)$. We obtain the blowup formula and show that the blowup function is given by a hyperelliptic σ-function with singular characteristic. We compute the contact terms and find, as a corollary, interesting identities between hyperelliptic Theta functions.

Keywords: Topological Field Theories, Integrable Hierarchies.

JHEP01(2001)004

Contents

1. Introduction

In the last few years, some work has been devoted to the study of an interesting interplay between supersymmetric gauge theories and their twisted counterparts, and integrable hierarchies (for a review, see [1, 2]). This interplay appears in a very natural way in the context of the so-called u-plane integral introduced by Moore and Witten [3], and has been explored in some detail in [4]–[8]. In a recent paper [9], we have extended and clarified the role of integrable systems in twisted $\mathcal{N} = 2$ theories by using the theory of hyperelliptic Kleinian functions. This allowed us to fully characterize the blowup function of Donaldson-Witten theory with gauge group SU(N) and, furthermore, to identify it as a τ function of a finite-gap solution (a multisoliton solution in the case of four-manifolds of simple type) of the KdV hierarchy. As a corollary of this result, we obtained a new expression for the contact terms $\mathcal{T}_{k,l}$ that appear in the low-energy twisted theory.

In this note we shall extend some of these results to the case of twisted theories with matter hypermultiplets. This case has been comparatively less studied except in the case of gauge group SU(2), which was considered in [3, 4, 10] and has been shown to lead to new results in four-manifold topology [11]. An interesting feature emerging in this framework, as shown in [3], is that SU(2) twisted theories with matter are only consistent on a general four-manifold if one turns on a magnetic flux. We shall give a generalization of this mechanism for gauge group $\mathrm{SU}(2\ell)$ and matter in the fundamental representation. As in the $\mathrm{SU}(2)$ case [3], this magnetic flux is related to the second Stiefel-Whitney class of the four-manifold.

We shall present some new results for the blowup formulae which are valid in any twisted theory with gauge group $\mathrm{SU}(2\ell)$ (with or without matter). In particular, we identify the blowup function as a hyperelliptic *fundamental* σ-function (whose characteristic is that of the vector of Riemann constants). The characteristic being singular, we show that the leading contribution to the blowup function is of order ℓ^2 in the "times" t_i, and we describe a procedure to expand it in terms of the vacuum expectation values of local observables of the twisted theory up to arbitrary order.

We finally derive novel expressions for the contact terms corresponding to descendant operators whose supporting two-cycles intersect. As a corollary of this analysis, we obtain a family of identities among hyperelliptic Θ-functions. The interpretation of these contact terms within the framework of the Whitham hierarchy leads to a new equation for the Seiberg-Witten effective prepotential.

2. Twisted theories with matter

In this section we shall consider the extension of the u-plane integral [3, 4, 5], in Donaldson-Witten theory with gauge group $\mathrm{SU}(N)$ when matter in the fundamental representation is included. We show the appearance of topological obstructions to define a monodromy invariant u-plane integral, and the way in which they can be overcome when the rank of the gauge group is odd. We conclude this section by presenting the blowup formula.

2.1 The u-plane integral

The u-plane integral gives the answer for the generating functional of twisted $\mathcal{N}=2$ $\mathrm{SU}(N)$ theories on four-manifolds X with $b_2^+=1$. The basic observables of the twisted theory are the Casimir operators of the gauge group,

$$\mathcal{O}_k = \frac{1}{k}\,\mathrm{Tr}\,\phi^k + \text{lower order terms}\,, \qquad k=2,\dots,N\,, \tag{2.1}$$

whose vacuum expectation values $u_k = \langle \mathcal{O}_k \rangle$ are gauge invariant coordinates for the Coulomb branch of the theory. Starting from the Casimirs, one can construct further (topological descendant) observables associated to two-cycles $S \in H_2(X)$ on the four-manifold,

$$I_k(S) = \frac{1}{k}\int_S \mathrm{Tr}\left(\phi^{k-1}F\right) + \cdots\,, \tag{2.2}$$

the dots standing for superpartner contributions. The generating functional of correlation functions in the twisted theory is then defined as:

$$Z(p_k, f_k, S) = \left\langle \exp\left[\sum_{k=2}^{N}\left(p_k\mathcal{O}_k + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i)\right)\right]\right\rangle, \tag{2.3}$$

where the S_i, $i = 1, \ldots, b_2(X)$, define a basis of $H_2(X)$.

The generating functional (2.3) can be explicitly evaluated by means of the exact low-energy effective action of the $\mathcal{N} = 2$ theory [12]. This action is encoded in a hyperelliptic curve $y^2 = f(x)$, the so-called Seiberg-Witten curve. As explained in [3, 5], for generic values of the hypermultiplet bare masses — such that there is no Higgs branch —, (2.3) has two contributions: one comes from the Coulomb branch, Z_{Coulomb}, and the other comes from the vanishing locus $\mathcal{D}$ of the discriminant Δ of the Seiberg-Witten curve. Z_{Coulomb} is given by an integral over the Coulomb branch, $\mathcal{M}_{\text{Coulomb}} = \mathbb{C}^{N-1} - \mathcal{D}$, the previously alluded u-plane integral. Its explicit expression turns out to be:

$$Z_{\text{Coulomb}} = \int_{\mathcal{M}_{\text{Coulomb}}} [dad\bar{a}]\ A(u_k)^{\chi} B(u_k)^{\sigma} e^{\sum p_k u_k + \sum f_{k,i} f_{l,j} \mathcal{T}_{k,l}(S_i, S_j)}\Psi\,. \tag{2.4}$$

The integrand of (2.4) has various ingredients. First of all, there is a gravitational measure first studied in [13] in the pure SU(2) case, generalized in [4, 5] to simply-laced groups, and further extended in [3] to the case of SU(2) with matter hypermultiplets. In the present case, the measure is given by:

$$A(u_k)^{\chi} = \alpha^{\chi}\left(\det\frac{\partial u_k}{\partial a^i}\right)^{\chi/2}, \qquad B(u_k)^{\sigma} = \beta^{\sigma}\Delta^{\sigma/8}\,, \tag{2.5}$$

where χ and σ are the Euler characteristic and the signature of X, while α and β are functions of the bare masses m_f and the dynamically generated scale Λ, that remain constant along $\mathcal{M}_{\text{Coulomb}}$, and whose determination might be achieved by comparison with the Donaldson-Witten theory at short distances. The first factor in (2.5) involves the determinant of the matrix of periods, whereas the second one contains the discriminant of the hyperelliptic curve. The factor Ψ is given by a complicated sum over a lattice Γ that involves a generalized Siegel-Narain theta function. A vector $\vec{\lambda}$ of this lattice is of the form

$$\vec{\lambda} = \vec{\lambda}_{\mathbb{Z}} + \vec{\xi}\,, \tag{2.6}$$

where

$$\vec{\lambda}_{\mathbb{Z}} = \sum_{i=1}^{N-1} \lambda_{\mathbb{Z}}^{i}\vec{\alpha}_i\,, \tag{2.7}$$

$\lambda^i_{\mathbb{Z}}$ are elements of $H^2(X, \mathbb{Z})$, $\vec{\alpha}_i$ are the simple roots of $\mathrm{SU}(N)$, and $\vec{\xi}$ is the magnetic flux that corresponds to the second Stiefel-Whitney class of the gauge bundle. For $\mathrm{SU}(N)$, it has the form [5]:

$$\vec{\xi} = \sum_{i=1}^{N-1} p^i \vec{w}_i \,, \tag{2.8}$$

where p^i are fixed elements of $H^2(X, \mathbb{Z})$ that represent a choice of the second Stiefel-Whitney class of the bundle $w_2(E)$, and $\vec{w}_i$ are the fundamental weights of $\mathrm{SU}(N)$. The function Ψ is the same appearing in the pure gauge case [3, 5]. The dependence on the effective coupling in this function has the form

$$\exp\left[-i\pi\overline{\tau}_{ij}(\lambda^i_+, \lambda^i_+) - i\pi\tau_{ij}(\lambda^i_-, \lambda^j_-)\right] . \tag{2.9}$$

In (2.9), $\lambda_+ = (\lambda, \omega)$ is the self-dual part of the two-form λ, constructed out of the unique anti-self-dual form $\omega \in H^{2,+}(X, \mathbb{R})$ such that $\omega^2 = 1$, by means of the usual $(\ ,\ )$ product in cohomology. Finally, Z_{Coulomb} contains contact terms $\mathcal{T}_{k,l}$ among the different two-observables. These contact terms have been studied in great detail in [3]–[9], and we will come back to them later on.

2.2 Monodromy invariance

As emphasized in [3, 5], in order to have a well-defined u-plane integral, the integrand of (2.4) should be *invariant* under the monodromies associated to the hyperelliptic curve. In the $\mathrm{SU}(2)$ theory with matter [3], this requirement leads to a nontrivial condition for the magnetic flux. Let us presently analyze the same problem for twisted theories with gauge group $\mathrm{SU}(N)$ and N_f massive hypermultiplets in the fundamental representation. The hyperelliptic curve corresponding to the infrared dynamics of the untwisted theory for $N_f \leq N$ is [14]

$$y^2 = P(\lambda, u_k)^2 - 4\Lambda^{2N-N_f} F(\lambda, m_f) \,, \tag{2.10}$$

where P is the characteristic polynomial of $\mathrm{SU}(N)$

$$P = x^N - \sum_{k=2}^{N} u_k x^{N-k} \,, \tag{2.11}$$

and F endowes the dependence on the bare masses,

$$F = \prod_{f=1}^{N_f} (x + m_f) \,. \tag{2.12}$$

The case $N < N_f < 2N$ can be similarly considered. We will show that, provided N is even, there is a particular value of the magnetic flux for which the u-plane integral is invariant under the semiclassical monodromies. Usually this is enough to guarantee the invariance under the strong coupling monodromies as well, so we will assume that this is the right choice to have a consistent u-plane integral.

The semiclassical monodromies that are specific of theories with matter, are those associated to elementary quarks becoming massless. They have the following form [15]:

$$a_{D,i} \longrightarrow a_{D,i} + (\vec{\mu}_I \otimes \vec{\mu}_I)_{ij} a^j \,, \tag{2.13}$$

where $\vec{\mu}_I$, $I = 1, \dots, N$, are the weights of the fundamental representation of $\mathrm{SU}(N)$. This monodromy corresponds to encircling the semiclassical singularity $\vec{a} \cdot \vec{\mu}_I + m_f = 0$. Under this monodromy, the effective couplings change as:

$$\tau_{ij} \longrightarrow \tau_{ij} + (\vec{\mu}_I)_i (\vec{\mu}_I)_j \,. \tag{2.14}$$

The monodromy introduces a phase in the u-plane integral, which can be computed as in [5]. The measure (2.5) contains a factor $\Delta^{\sigma/8}$. Going around the elementary quark singularity one picks a phase $\exp(\pi i \sigma/4)$. The other phase comes from the contribution (2.9) to the Siegel-Narain theta function, as a consequence of the shift in the effective gauge couplings under the monodromy,

$$\exp\Big[- i\pi \Big(\vec{\lambda} \cdot \vec{\mu}_I, \vec{\lambda} \cdot \vec{\mu}_I \Big) \Big] \,. \tag{2.15}$$

This has to cancel against the overall phase coming from the measure. In particular, the phase (2.15) should be independent of $\vec{\lambda}$. It is easy to check that one has indeed the desired cancellation if N is *even* and the magnetic flux is given by

$$\vec{\xi} = w_2(X) \vec{\rho} \,, \tag{2.16}$$

where $\vec{\rho}$ is the Weyl vector (i.e. the sum of the fundamental weights), and $w_2(X)$ is the second Stiefel-Whitney class of the four-manifold X. Notice that for SU(2) one obtains, in the root basis, that $\xi_1 = w_2(X)/2$, which is precisely the magnetic flux found in [3] for theories with massive matter. To verify that (2.16) guarantees the cancellation, one has to use Wu's formula (which states that $(w_2(X), \alpha) \equiv \alpha^2 \bmod 2$ for any two-cohomology class), the fact that $w_2^2(X) \equiv \sigma \bmod 8$, and that

$$\vec{\mu}_I \cdot \vec{\rho} = \frac{1}{2}(N - 2I + 1) \,. \tag{2.17}$$

We will assume from now on that N is even, $N = 2\ell$. It may be possible that there is a choice of $\vec{\xi}$ which makes the theory well defined also for N odd, but we have not found any. Notice that the need to choose a flux reflects the fact that in the twisted theory there are fields which are sections of the bundle $S^+ \otimes E$, where S^+ is the spinor bundle on X and E is the gauge bundle. In general, this product bundle does not exist, and this is what requires the choice of a nontrivial Stiefel-Whitney class for the gauge bundle. It would be interesting to understand (2.16) from this point of view, and this might give a hint of how to make the choice of $\vec{\xi}$ for N odd (or prove that there is none).

2.3 Blowup formula

Using this information we can already compute the blowup formula as in [3, 4, 5]. Consider the four-manifold $\widehat{X}$, obtained from X by blowing up a point p, $\widehat{X} = \mathrm{Bl}_p(X)$. This means that there is a map $\pi : \widehat{X} \to X$ that is the identity everywhere except at $B = \pi^{-1}(p)$, where $B \in H_2(\widehat{X})$ such that $B^2 = -1$. B is called the class of the exceptional divisor. The anti-self-dual two-form contribution to the u-plane integral in $\widehat{X}$ gets modified to $\widehat{\lambda}^i_- = \lambda^i_- + n^i B$ with $n^i \in \mathbb{Z}$. Up to now, we have considered the blowup formula only when there is no magnetic flux through the exceptional divisor. As noticed in [3] for the SU(2) case, the choice (2.16) actually forces us to shift the flux by $B\vec{\rho}$. This is due to the fact that $w_2(\widehat{X}) = w_2(X) + B$ (mod 2). Thus, vectors $\widehat{\vec{\lambda}}$ of the lattice $\widehat{\Gamma}$ corresponding to the u-plane integral of the blownup four-manifold have the form

$$\widehat{\vec{\lambda}} = \vec{\lambda} + (\vec{\alpha} + \vec{\rho})B = \vec{\lambda} + \sum_{i=1}^{N-1} \left(n^i + \sum_{j=1}^{N-1} (C^{-1})_j{}^i \right) \vec{\alpha}_i B \,, \tag{2.18}$$

C being the Cartan matrix. This amounts to the appearance of a particular characteristic $\sum_{j=1}^{N-1}(C^{-1})_j{}^i$ in the factor (2.9), which is easily seen to be integer (half-integer) for even (odd) i. Thus, the $\vec{\beta}$ characteristic of the blowup function becomes

$$\vec{\beta} = \vec{\Sigma} = \left(\frac{1}{2}, 0, \frac{1}{2}, \ldots, 0, \frac{1}{2} \right) . \tag{2.19}$$

The $\vec{\alpha}$ characteristic is the same as in the pure gauge case

$$\vec{\alpha} = \vec{\Delta} = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2} \right) , \tag{2.20}$$

so that, finally, the blowup function of the massive twisted theory with gauge group SU(2ℓ) has a half-integer characteristic, which is even (odd) for even (odd) values of ℓ. Aside of this important aspect, the derivation of the blowup formula within the u-plane integral follows the same lines developed in [3, 5]. In particular, there is a contribution from the measure due to the fact that both the Euler characteristic and the signature of the manifold are shifted by the blowup. The final outcome is that the generating functional (2.3) for the blownup four-manifold can be written as

$$\begin{aligned} \widehat{Z}(p_k, f_k, t_k, S) &= \left\langle \exp \left[\sum_{k=2}^{N} \left(p_k \mathcal{O}_k + t_k I_k(B) + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i) \right) \right] \right\rangle_{\widehat{X},\vec{\Sigma}} \\ &= \left\langle \exp \left[\sum_{k=2}^{N} \left(p_k \mathcal{O}_k + \sum_{i=1}^{b_2(X)} f_{k,i} I_k(S_i) \right) \right] \tau_{\vec{\Sigma}}(t_k|\mathcal{O}_k) \right\rangle_X , \end{aligned} \tag{2.21}$$

where the blowup function $\tau_{\vec{\Sigma}}(t_i|u_k)$ is given by

$$\tau_{\vec{\Sigma}}(t_i|u_k) = \left(\det \frac{\partial u_k}{\partial a^j} \right)^{1/2} \Delta^{-1/8} \exp \left\{ - \sum_{k,l=2}^{N} t_k t_l \mathcal{T}_{k,l} \right\} \Theta_{[\vec{\Delta},\vec{\Sigma}]}(\vec{z}|\tau) \,, \tag{2.22}$$

up to an overall factor which only depends on m_f and Λ. The argument of the Θ-function in (2.22) is

$$z_i = \sum_{k=2}^{N} \frac{t_k}{2\pi} \frac{\partial u_k}{\partial a^i} \,. \tag{2.23}$$

As in the case of pure $\mathcal{N}=2$ Yang-Mills, the blowup can be interpreted as a local defect, and we expect the blowup function to be given by an expansion of the form

$$\tau_{\vec{\Sigma}}(t_i|u_k) = \sum_{n\geq 0} \sum_{i_1,\ldots,i_n} t_{i_1}\cdots t_{i_n} \mathcal{B}^{(n)}_{i_1\cdots i_n}(u_k, m_f, \Lambda) \,, \tag{2.24}$$

where $\mathcal{B}^{(n)}_{i_1\cdots i_n}(u_k, m_f, \Lambda)$ are homogeneous polynomials, the degree of their variables being given by their mass dimensions.

3. The blowup function

In this section we shall study in detail many aspects of the blowup function corresponding to twisted theories with matter. We first identify the fixed characteristic of the Θ-function as a singular one. This fact affects significantly the behaviour of the blowup function and, consequently, modifies the generic expression of the contact terms with respect to that of the pure gauge case, the latter being given by [4]

$$\mathcal{T}_{k,l} = -\frac{1}{2\pi i} \frac{\partial u_k}{\partial a^i} \frac{\partial u_l}{\partial a^j} \frac{\partial}{\partial \tau_{ij}} \log \Theta_{[E]}(0|\tau) \,, \tag{3.1}$$

where $[E]$ is a (non-singular) even half-integer characteristic, $[E] = [\vec{\Delta}, \vec{0}]$. We further show how the methods of [9] can be extended to actually compute the polynomials $\mathcal{B}^{(n)}_{i_1\cdots i_n}(u_k, m_f, \Lambda)$ up to arbitrary order.

3.1 Vanishing properties of Θ-functions

In contrast to what happens in the pure gauge case, the blowup function in the twisted theory with matter hypermultiplets has a fixed — and very peculiar — characteristic (2.19) and (2.20). It is well known from the theory of Riemann surfaces (see for example [16, 17]) that characteristics can be associated to the branch points of the curve through the elements of the jacobian constructed out of the Abel map. More concretely, consider a hyperelliptic curve Σ_g of genus g and a basis of homology cycles $(A^i, B_j) \in H_1(\Sigma_g, \mathbb{Z})$ with its respective normalized holomorphic differentials $d\omega_k$. Let e_α, $\alpha = 1, \ldots, 2g+2$ be the branch points of the surface, and take e_1 as a reference point. We can now define $2g+2$ vectors in the jacobian $\vec{\mathcal{U}}_\alpha$ as the image of the divisors $\mathcal{D}_\alpha = e_\alpha - e_1$ under the Abel map

$$\vec{\mathcal{U}}_\alpha = \frac{1}{2\pi i} \int_{e_1}^{e_\alpha} d\vec{\omega} = \vec{\epsilon}_\alpha + \tau \vec{\delta}_\alpha \,, \tag{3.2}$$

and the corresponding characteristic is $[\mathcal{U}_\alpha] = 1/2[\vec{\epsilon}_\alpha, \vec{\delta}_\alpha]$.

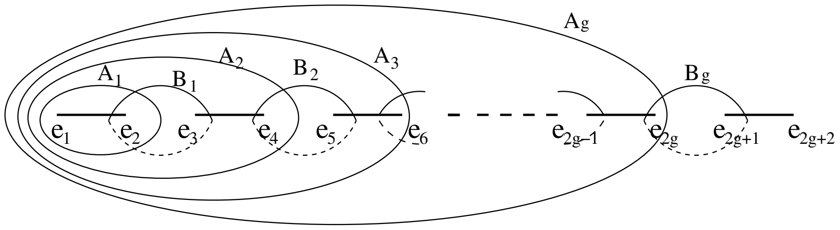

Figure 1: Symplectic basis of homology cycles.

There is a one to one identification among half-integer characteristics and partitions of the branch points into two groups $I = I_m \cup (I/I_m)$, where $I_m = \{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_{g+1-2m}}\}$. It is given by

$$[\Upsilon] = \sum_{k=1}^{g+1-2m} [\mathcal{U}_{\alpha_k}] + [K] \pmod 1\,, \tag{3.3}$$

where $[K]$ is the characteristic corresponding to the vector of Riemann constants $\vec{K}$ (which, in the case of a hyperelliptic curve, is a half-period). For example, using the basis of cycles shown in figure 1, the characteristic $[E]$ in (3.1) is the one corresponding to the natural partition of branch points in the pure gauge Seiberg-Witten solution. Finally, the characteristic of the vector of Riemann constants in such basis,

$$[K] = \left[\vec{\Delta}, \vec{\Sigma}\right], \tag{3.4}$$

turns out to be precisely the half-integer characteristic appearing in the blowup function of twisted theories with matter. For hyperelliptic curves of genus $g > 2$, this characteristic is *singular* having a zero of order $(g+1)/2$ or $g/2$ at the origin, provided g is respectively odd or even. Then, the expansion (2.24) of the blowup function in twisted theories with matter and gauge group SU(2ℓ) takes the form

$$\tau_{\vec{\Sigma}}(t_i|u_k) = \sum_{n\geq \ell} \sum_{i_1,\ldots,i_n} t_{i_1}\cdots t_{i_n} \mathcal{B}^{(n)}_{i_1\cdots i_n}(u_k, m_f, \Lambda)\,. \tag{3.5}$$

Notice that, in contrast to the case with no flux, this blowup function vanishes at $t_i \to 0$.

3.2 The hyperelliptic fundamental σ-function

In order to further characterize the blowup function of twisted theories with matter, let us introduce shortly some algebro-geometrical ingredients. Given a hyperelliptic curve $y^2 = f(x)$ and a basis of abelian differentials of the first kind $dv_k = x^{g-k}dx/y$, one can compute the period integrals

$$A^i{}_k = \frac{1}{2\pi i}\oint_{A^i} dv_k\,, \qquad B_{ik} = \frac{1}{2\pi i}\oint_{B_i} dv_k\,, \tag{3.6}$$

in terms of which we can define the period matrix as

$$\tau_{ij} = B_{ik}(A^{-1})^k{}_j \,. \tag{3.7}$$

The low-energy $\mathcal{N} = 2$ theory with N_f matter hypermultiplets in the fundamental representation is described by a prepotential $\mathcal{F}(a^i, m_f, \Lambda)$, where

$$a^i(u_k, \Lambda) = \frac{1}{2\pi i} \oint_{A^i} \frac{xW'(x)}{\sqrt{W^2(x) - 4\Lambda^{2N-N_f}}} dx \,, \tag{3.8}$$

with $W = P/\sqrt{F}$, P and F being given in (2.11) and (2.12). The same expression holds for the dual variables $a_{D,i} \equiv \partial\mathcal{F}/\partial a^i$, with B_i instead of A^i. The effective gauge couplings are given by (3.7), and

$$A^i{}_k = \frac{\partial a^i}{\partial u_{k+1}} \,, \qquad B_{ik} = \frac{\partial a_{D,i}}{\partial u_{k+1}} \,. \tag{3.9}$$

To construct a hyperelliptic σ-function, we also need a basis of abelian differentials of the second kind $dr^k(x)$. It is provided by means of a Weierstrass polynomial $F(x_1, x_2)$, through the following identity [18]:

$$\sum_{k=1}^{g} dv_k(x_1)\, dr^k(x_2) = -\frac{1}{2y_1} \frac{\partial}{\partial x_2} \Big(\frac{y_2}{x_1 - x_2} \Big) dx_1\, dx_2 + \frac{F(x_1, x_2)}{4(x_1 - x_2)^2} \frac{dx_1\, dx_2}{y_1 y_2} \,. \tag{3.10}$$

Then, we define the following matrices of η-periods:

$$\eta^{ki} = -\frac{1}{2\pi i} \oint_{A^i} dr^k \,, \qquad \eta'^k{}_i = -\frac{1}{2\pi i} \oint_{B_i} dr^k \,, \tag{3.11}$$

that obey the Legendre relation

$$\eta = 2\kappa A \,, \qquad \eta' = 2\kappa B - \frac{1}{2}(A^{-1})^t \,, \tag{3.12}$$

where κ is a symmetric matrix that, of course, depends on F. These ingredients are enough to define the *hyperelliptic fundamental σ-function* by the formula [18]:

$$\sigma_f^F(\vec{v}) = C^{-1} \exp\left\{ v_i \kappa^{il} v_l \right\} \Theta_{[K]}\Big((2\pi i)^{-1} v_l (A^{-1})^l{}_i | \tau \Big) \,, \tag{3.13}$$

where C is constant (with respect to the v_l) and it is given by

$$C = i^\ell (\det A)^{1/2} \Delta^{1/8} \,. \tag{3.14}$$

It is now immediate to see that, after the identification $v_l \equiv i t_{l+1}$, this is nothing but the blowup function of $\mathcal{N} = 2$ twisted theories with gauge group SU(2ℓ) and $N_f < 4\ell$ matter hypermultiplets, i.e.

$$\tau_{\vec{\Sigma}}(t_i | u_k) = i^\ell \sigma_f^F(v_l = i t_{l+1}) \,. \tag{3.15}$$

The overall factor i^ℓ in (3.14) and (3.15) is choosen just for convenience.

JHEP01(2001)004

The contact terms are given by the matrix κ (notice that their transformation properties under the action of the modular group $\mathrm{Sp}(2g,\mathbb{Z})$, given in [18] and [19], is indeed the same) for a given Weierstrass polynomial still to be determined. In the pure gauge theory with magnetic fluxes turned off, the semiclassical vanishing of the contact terms allows for an analytical determination of F. We shall study this issue in presence of matter in the next section and see that there is no such a simplification and, in turn, comparison with semiclassical results would become necessary. It is somehow an expected result that the blowup function be a σ-function, as long as the main property of the latter is its invariance under the action of the modular group.

3.3 Expansion of the blowup function

The explicit expansion of the blowup function can be done following the method introduced in [9], which is based in a series of developments carried out by Oskar Bolza one century ago [20]. The starting point is a partial differential equation for σ_f^F with respect to a branch point e_α of Σ_g,[1] that can be written for any genus [20],

$$\frac{\partial\sigma_f^F}{\partial e_\alpha}+\sigma_f^F\frac{\partial\log C}{\partial e_\alpha}=-\sum_{i,j=1}^{g}\left\{p_{ij}^F(e_\alpha)v_i\frac{\partial\sigma_f^F}{\partial v_j}+\frac{1}{2}\sigma_f^F q_{ij}^F(e_\alpha)v_iv_j-\right.$$
$$\left.-\frac{e_\alpha^{2g-i-j}}{f'(e_\alpha)}\left(\frac{\partial^2\sigma_f^F}{\partial v_i\partial v_j}-2\kappa_{ij}\sigma_f^F\right)\right\}, \qquad (3.16)$$

where the matrices $p_{ij}^F(e_\alpha)$ and $q_{ij}^F(e_\alpha)$ are given by

$$\sum_{i,j=1}^{g}p_{ij}^F(e_\alpha)x^{g-i}h_j(z)=\frac{1}{2}\frac{(x-z)^{g-1}}{x-e_\alpha}-\frac{1}{2}\frac{(e_\alpha-z)^{g-1}}{f'(e_\alpha)}\frac{F(x,e_\alpha)}{(x-e_\alpha)^2},$$
$$\sum_{i,j=1}^{g}q_{ij}^F(e_\alpha)x^{g-i}z^{g-j}=\frac{1}{8}\left(\frac{1}{x-e_\alpha}+\frac{1}{z-e_\alpha}\right)\frac{F(x,z)}{(x-z)^2}+\frac{1}{4}\frac{1}{(x-z)^2}\frac{\partial F(x,z)}{\partial e_\alpha}-$$
$$-\frac{1}{8}\frac{F(x,e_\alpha)F(z,e_\alpha)}{f'(e_\alpha)(x-e_\alpha)^2(z-e_\alpha)^2}, \qquad (3.17)$$

the $'$ denoting derivatives w.r.t. x, and the function $h_j(z)$ being implicitely defined through the relation

$$(x-z)^{g-1}=\sum_j x^{g-j}h_j(z). \qquad (3.18)$$

If we plug in the Taylor expansion of σ_f^F in (3.16),

$$\sigma_f^F(\vec{v})=\sum_{n=\ell}^{\infty}\varsigma_n(\vec{v}), \qquad (3.19)$$

where $\{\varsigma_n(\vec{v})\}$ are homogeneous polynomials of degree n in v_l (notice that the sum

[1] As explained in [9], there is a well-defined procedure to trade this sort of derivatives, which are of little practical use, for a differential equation involving v_l-derivatives and the coefficients of the curve.

runs over even or odd integers according to the parity of ℓ), a set of recursive relations shows up immediately:

$$\begin{aligned}\frac{\partial\varsigma_{n-2}}{\partial e_\alpha} + \varsigma_{n-2}\frac{\partial \log C}{\partial e_\alpha} = -\sum_{i,j=1}^{g} &\left\{ p_{ij}^F(e_\alpha) v_i \frac{\partial \varsigma_{n-2}}{\partial v_j} + \frac{1}{2} q_{ij}^F(e_\alpha) v_i v_j \varsigma_{n-4} - \right. \\ &\left. - \frac{e_\alpha^{2g-i-j}}{f'(e_\alpha)}\left(\frac{\partial^2\varsigma_n}{\partial v_i\partial v_j} - 2\kappa_{ij}\varsigma_{n-2}\right)\right\}.\end{aligned} \tag{3.20}$$

We already know that the leading term in σ_f^F at the origin is of order ℓ. Then, setting $n = \ell$ in (3.20), we obtain

$$\sum_{i,j=1}^{g} e_\alpha^{2g-i-j}\frac{\partial^2\varsigma_\ell}{\partial v_i\partial v_j} = 0\,, \tag{3.21}$$

for any branch point e_α. The solution to this equation is provided by the determinant of the Hankel matrix

$$\varsigma_\ell = \det H = \det\begin{pmatrix} v_1 & v_2 & \dots & v_\ell \\ v_2 & v_3 & \dots & v_{\ell+1} \\ \vdots & \vdots & \dots & \vdots \\ v_\ell & v_{\ell+1} & \dots & v_{2\ell-1}\end{pmatrix}. \tag{3.22}$$

The overall multiplicative factor has been fixed by comparison with the result obtained in [18] following a different approach. Then, the leading term of the blowup function is given by i^ℓ times the determinant of the Hankel matrix with $v_l \to it_{l+1}$. This is the answer for $\mathcal{B}^{(\ell)}_{i_1\cdots i_\ell}(u_k, m_f, \Lambda)$. Notice that the constant C is nothing but

$$C = \left.\frac{\partial^\ell\Theta_{[K]}\big((2\pi i)^{-1}v_l(A^{-1})^l{}_i|\tau\big)}{\partial v_1\partial v_3\cdots\partial v_{2\ell-1}}\right|_{\vec{v}=\vec{0}}. \tag{3.23}$$

Being given by the derivative of a Θ-function, the derivative of $\log C$ with respect to e_α can be treated again by means of the formalism developed by Bolza [20] resulting in

$$\frac{\partial \log C}{\partial e_\alpha} = \sum_{i,j=1}^{g}\frac{e_\alpha^{2g-i-j}}{f'(e_\alpha)}\left\{\left.\frac{\partial^{\ell+2}\sigma_f^F}{\partial v_i\partial v_j\partial v_1\cdots\partial v_{2\ell-1}}\right|_{\vec{v}=\vec{0}} - 2\kappa_{ij}\right\} - \sum_{i=1}^{g} p_{ii}^F(e_\alpha)\,. \tag{3.24}$$

Then, the recursive relations (3.20) are

$$\begin{aligned}\frac{\partial\varsigma_{n-2}}{\partial e_\alpha} = &\sum_{i=1}^{g} p_{ii}^F(e_\alpha)\varsigma_{n-2} - \\ &-\sum_{i,j=1}^{g}\left\{p_{ij}^F(e_\alpha)v_i\frac{\partial\varsigma_{n-2}}{\partial v_j} + \frac{1}{2}q_{ij}^F(e_\alpha)v_iv_j\varsigma_{n-4} - \right. \\ &\left. - \frac{e_\alpha^{2g-i-j}}{f'(e_\alpha)}\left(\frac{\partial^2\varsigma_n}{\partial v_i\partial v_j} - \left.\frac{\partial^{\ell+2}\sigma_f^F}{\partial v_i\partial v_j\partial v_1\cdots\partial v_{2\ell-1}}\right|_{\vec{v}=\vec{0}}\varsigma_{n-2}\right)\right\}.\end{aligned} \tag{3.25}$$

Notice that the coefficient of the last term in the third line only depends on ς_ℓ and $\varsigma_{\ell+2}$ so that recursivity is safe.

The result obtained above is nontrivial in the sense that it is not dictated *a priori* by symmetry arguments or the semiclassical behaviour. Provided the Weierstrass polynomial is obtained for a given gauge group and matter content, the recursive rule (3.20) would allow for a computation of the blowup function up to arbitrary order in time variables. Finally, once the blowup function is recognized to be a hyperelliptic σ-function, it is immediate to show that it is the τ-function of a finite gap solution of the KdV hierarchy. In fact, the proof is exactly the same given in [9] for the pure gauge theory.

4. Contact terms

As originally noticed in [4], an explicit expression for the contact terms can be derived from the blowup function by requiring invariance under $\mathrm{Sp}(2g, \mathbb{Z})$ duality transformations, and taking into account that they must vanish semiclassically [3]. We shall perform a similar analysis in what follows. It is instructive to consider first the case of SU(2), where everything can be written in terms of elementary elliptic functions. In this case, the blowup function (2.22) is given by

$$\tau_{1/2}(t_i|u_k) = h^{-1/2}\Delta^{-1/8}\mathrm{e}^{-t^2\mathcal{T}_{2,2}}\vartheta_1\left((2\pi h)^{-1}t|\tau\right), \tag{4.1}$$

where $h = da/du$. Using Thomae's formula, and choosing the appropriate normalization, one finds that $\Delta^{1/8} = (1/2)h^{-3/2}\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau)$. Using the identity $\vartheta_1'(0|\tau) = -\pi\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau)$, the expansion of (4.1) in t turns out to be

$$\tau_{1/2}(t_i|u_k) = -t\left\{1 - t^2\left(\mathcal{T}_{2,2} - \frac{1}{24\pi^2h^2}\frac{\vartheta_1'''(0|\tau)}{\vartheta_1'(0|\tau)}\right) + \mathcal{O}\left(t^4\right)\right\}. \tag{4.2}$$

It follows from (3.5) that the contact term is

$$\mathcal{T}_{2,2} = \frac{1}{24\pi^2h^2}\frac{\vartheta_1'''(0|\tau)}{\vartheta_1'(0|\tau)} + \mathcal{B}^{(3)}(u, m_f, \Lambda). \tag{4.3}$$

As discussed above, $\mathcal{B}^{(3)}(u, m_f, \Lambda)$ is such that the contact terms vanish semiclassicaly. Taking into account the identity

$$\frac{\vartheta_1'''(0|\tau)}{\vartheta_1'(0|\tau)} = -\pi^2E_2(\tau), \tag{4.4}$$

where $E_2(\tau)$ is the normalized Eisenstein series, we see that (4.3) is indeed consistent with the explicit expression for the contact terms derived in [3, 4, 10]:

$$\mathcal{T}_{2,2} = -\frac{1}{24h^2}E_2(\tau) + \frac{1}{3}\left(u + \delta_{N_f,3}\frac{\Lambda^2}{64}\right), \tag{4.5}$$

and from this equation we can also read the explicit value of $\mathcal{B}^{(3)}(u, m_f, \Lambda)$. Now, as mentioned earlier, the contact term $\mathcal{T}_{2,2}$ in such case is given by (3.1) with $k = l = 2$. Thus, the following identity emerges:

$$-\frac{1}{2\pi i}\left(\frac{du}{da}\right)^2 \partial_\tau \log\vartheta_4(0|\tau) = \frac{1}{24\pi^2}\left(\frac{du}{da}\right)^2 \frac{\vartheta_1'''(0|\tau)}{\vartheta_1'(0|\tau)} + \frac{u}{3}\,. \tag{4.6}$$

This result can be shown analytically by means of the theory of elliptic functions [4]. As we will see in what follows, different expressions for the contact terms lead to generalizations of this sort of identities to higher genus curves.

Let us first introduce a short notation for the contraction of derivatives of the singular Θ-function and the inverse of the A^i-periods of the Seiberg-Witten differential

$$\vartheta^{(n)}_{i_1\cdots i_n} \equiv \frac{1}{(2\pi)^n n!}\frac{\partial u_{i_1}}{\partial a^{j_1}}\cdots\frac{\partial u_{i_n}}{\partial a^{j_n}}\frac{\partial^n\Theta_{[K]}(0|\tau)}{\partial z_{j_1}\cdots\partial z_{j_n}}\,. \tag{4.7}$$

Now, taking into account our earlier results, the expansion of $\Theta_{[K]}(\vec{z}\mid\tau)$ reads

$$\Theta_{[K]}(\vec{z}\mid\tau) = t_{i_1}\cdots t_{i_\ell}\vartheta^{(\ell)}_{i_1\cdots i_\ell} + t_{i_1}\cdots t_{i_{\ell+2}}\vartheta^{(\ell+2)}_{i_1\cdots i_{\ell+2}} + \cdots\,, \tag{4.8}$$

where repeated indices are summed. If we now expand the blowup function (2.22) and we compare the result with the structure of (3.5), we find the following equations for the contact terms:

$$\frac{i^\ell}{C}\vartheta^{(\ell)}_{(i_1\cdots i_\ell}\mathcal{T}_{mn)} = \frac{i^\ell}{C}\vartheta^{(\ell+2)}_{i_1\cdots i_\ell mn} - \mathcal{B}^{(\ell+2)}_{i_1\cdots i_\ell mn}(u_k, m_f, \Lambda)\,, \tag{4.9}$$

where the indices in the l.h.s. of the equation are symmetrized. This is, in principle, the generalization of (4.3) to the higher rank case. One can also obtain a much more explicit equation for $\mathcal{T}_{2,2}$ by putting $t_{i>2}\equiv 0$, and considering only terms in the blowup function depending on t_2. Since $\vartheta^{(\ell)}_{2\cdots 2}$ *vanishes* due to the fact that the determinant of the Hankel matrix has no v_1^ℓ term, we need to attain higher orders in the expansion of the blowup function. The first nonvanishing term in the expansion is the $v_1^{\ell^2}$ term. This is due to the fact that σ^F is a homogeneous function of degree $-\ell^2$ provided we assign a weight equal to $-i$ for v_i. The expression for the contact term $\mathcal{T}_{2,2}$ arising from this analysis is

$$\mathcal{T}_{2,2} = \frac{\vartheta^{(\ell^2+2)}_{2\cdots 2}}{\vartheta^{(\ell^2)}_{2\cdots 2}} - \frac{\mathcal{B}^{(\ell^2+2)}_{2\cdots 2}(u_k, m_f, \Lambda)}{\mathcal{B}^{(\ell^2)}_{2\cdots 2}(u_k, m_f, \Lambda)}\,. \tag{4.10}$$

For the general case of $t_{i>2}\neq 0$, (4.9) gives different expressions for the contact terms involving derivatives of higher Casimir operators.

To illustrate the above results, consider the case of pure gauge SU(4) theory. The Weierstrass function that appears in the σ-function has the form [9]

$$F = Q(x_1)R(x_2) + Q(x_2)R(x_1)\,, \tag{4.11}$$

where the polynomials Q and R are given by

$$Q(x) = P(x) + 2\Lambda^4\,, \qquad R(x) = P(x) - 2\Lambda^4\,. \tag{4.12}$$

Using the differential equations of section 3, one obtains the expansion

$$\begin{aligned}\sigma_f^F &= t_3^2 - t_2 t_4 - \frac{1}{12}t_2^4 + \frac{u_2}{6}t_4 t_2^3 - \frac{u_3}{3}t_2 t_3^3 + \frac{u_3}{2}t_2^2 t_3 t_4 - \frac{u_2 u_3}{6}t_3^3 t_4 - \frac{u_3^2}{8}t_3^2 t_4^2 + \frac{u_4}{2}t_2^2 t_4^2 - \\ &- \frac{u_3 u_4}{6}t_3 t_4^3 - \frac{u_2^2+4u_4}{12}t_3^4 + \frac{u_3^2 - 4u_2 u_4}{24}t_2 t_4^3 + \frac{4\Lambda^8 - u_4^2}{12}t_4^4 + \frac{u_2}{180}t_2^6\,. \end{aligned} \tag{4.13}$$

We then find $\mathcal{B}^{(6)}_{2\cdots 2}/\mathcal{B}^{(4)}_{2\cdots 2} = -u_2/15$. One can then check (4.10) by using the semi-classical expansion of the effective gauge coupling up to order Λ^8. This result for the quotient of the $\mathcal{B}$'s also holds in the $N_f = 1$ and $N_f = 2$ theories. As a further check, one finds that for SU(6) $\mathcal{B}^{(11)}_{2\cdots 2}/\mathcal{B}^{(9)}_{2\cdots 2} = -\frac{3}{70}u_2$.

The expressions obtained above for the contact terms are valid for any number of massive hypermultiplets. In particular, they apply when N_f is an even number and the bare masses are degenerated in pairs, $m_f = m_{f+N_f/2}$. In such cases, the results for the contact terms must coincide with those carried out earlier in [19]. For example, comparison of $\mathcal{T}_{2,2}$ implies the following identity between hyperelliptic Θ-functions,

$$-\frac{1}{2\pi i}\frac{\partial u_2}{\partial a^i}\frac{\partial u_2}{\partial a^j}\frac{\partial}{\partial \tau_{ij}}\log\Theta_{[E]}(0|\tau) = \frac{\vartheta^{(\ell^2+2)}_{2\cdots 2}}{\vartheta^{(\ell^2)}_{2\cdots 2}} - \frac{\mathcal{B}^{(\ell^2+2)}_{2\cdots 2}(u_k, m_f, \Lambda)}{\mathcal{B}^{(\ell^2)}_{2\cdots 2}(u_k, m_f, \Lambda)}\,. \tag{4.14}$$

Similar expressions can be obtained in principle from (4.9).

The new expressions for the contact terms found in the present paper should be useful to compute the instanton corrections to the effective prepotential of $\mathcal{N} = 2$ supersymmetric theories with arbitrary matter content. Since the second derivative of the prepotential $\partial^2\mathcal{F}/\partial\Lambda^2$ is (up to a numerical constant) equal to $\mathcal{T}_{2,2}$, and this is given by (4.10), one can follow the arguments in [8, 21, 22] to obtain the semiclassical expansion of $\mathcal{F}$ through a recursive procedure.

Acknowledgments

The work of J.D.E. has been supported by the Argentinian National Research Council (CONICET) and by a Fundación Antorchas grant under project number 13671/1-55. The work of M.M. has been supported by DOE grant DE-FG02-96ER40959.

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